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Signatures of finite representation of real, simple Lie algebras

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Abstract. The paper deals with the arbitrary irreducible representations of simple Lie algebras of types G_2, F_4, B_r, C_r, D_r and the Hermitian forms being invariant relative to this representation. Formulas and tables for calculating Hermitian forms signatures are obtained.

1. Introduction

Consider an irreducible reprepresentation $\varphi: g \to sl(V)$ of simple complex Lie algebra g. Let λ be the highest weight, and let χ_λ be the character of the representation φ. Denote by g_σ any real form of inner type for the algebra g. From [1] it follows that $\varphi(g_\sigma) \subset su(p, q)$, where $p+q = \dim V$. Let $\delta = p - q$. Formulas for calculation of δ in the case of simple classical Lie algebras were obtained in [1]. Formulas for $|\delta|$ in the case of real algebras $G, FI, FII, so(p, q)$ where obtained in [2] and [3]. Finite $su(p, q)$ representations were considered in [4]; in particular, the formulas for δ in the case of algebras $su(1, 1)$ $su(2, 1)$, $su(2, 2)$, $su(3, 1)$ where obtained there.

2. Definitions

Let g_τ be the fixed compact real form of the algebra g, and let τ be the conjugation of the algebra g with respect to g_τ. Consider an involution θ of the algebra g such that $\theta(g_\tau) = g_\tau$. Then $g_\tau = (1+\theta)g_\tau + (1-\theta)g_\tau$. Let $\sigma = \tau \circ \theta = \theta \circ \tau$, $g_\sigma = (1+\theta)g_\tau + \sqrt{-1}(1-\theta)g_\tau$. Therefore g_σ is a real form of the algebra g, and σ is a conjugation of the algebra g with respect to g_σ. The real form is called the real form of inner type, if θ is an inner automorphism of g_τ, that is $\theta \in \mathrm{Int}(g_\tau)$. Suppose t is a Cartan subalgebra of the algebra g_τ such that $\theta(t) = t$, h is a Cartan subalgebra of g such that $t^{\mathbb{C}} = h$, R is a root system associated with the pair (g, h). Let $B(,)$ be a Killing form of g, and let

$$(,) = -\frac{1}{(2\pi)^2} B(,)$$

be a positive definite scalar product on t. Let $\alpha \in R$; by H_α denote an element of h such that $B(H_\alpha, H) = \alpha(H)$ for all $H \in h$. Define the embedding $\psi: R \to t$ by $\psi(\alpha) = 2\pi\sqrt{-1}$

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H_α for all $\alpha \in R$. Suppose $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ is a set of the simple roots of R, $\{H_i\}_{i=1}^r$ is a basis of t such that $(H_i, \alpha_j) = \delta_{ij}$, $i, j = 1, \ldots, r$. If $\theta \in \mathrm{Int}(g_\tau)$, then without loss of generality $\theta = \exp(\mathrm{ad}(H_{i_0}/2))$ for some i_0, $1 \leq i_0 \leq r$ [5]. Let $R^\vee$ be the root system dual to R, that is

$$R^\vee = \left\{ \frac{2\alpha}{(\alpha, \alpha)} \middle| \alpha \in R \right\}.$$

Suppose W is a Weyl group of R, $P(R^\vee)$ is a group of weights for $R^\vee$ [6], and $W'_a = W * P(R^\vee)$ is a semidirect product of the groups W and $P(R^\vee)$.

3. The formula for $|\delta|$

From [1] it follows that $\varphi(g_\sigma) \subset su(p, q)$ if and only if there exists a linear operator A on V such that

$$\varphi(\theta(x)) = A^{-1}\varphi(x)A \qquad \forall x \in g \tag{1}$$

and $A^2 = 1$. Furthermore, $|\mathrm{Tr}\, A| = |\delta|$. From [2] it follows that if $\theta = \exp(\mathrm{ad}(H_{i_0}/2))$, then there exists a weight vectors basis of g-module V such that A is diagonal in it. Furthermore

$$|\delta| = |\mathrm{Tr}\, A| = |\chi_\lambda(H_{i_0}/2))| \tag{2}$$

where χ_λ is the character of the irreducible representation φ. Consider a function

$$A_{\lambda+\rho}(H) = \sum_{s \in W} \det s \exp(2\pi\sqrt{-1}(s(\lambda+\rho), H))$$

where

$$\rho = \frac{1}{2} \sum_{\beta \in R, \beta > 0} \beta$$

is half the sum of the positive roots R, and W is the Weyl group. Then [5]

$$A_\rho(H) = (2\sqrt{-1})^l \prod_{\beta \in R, \beta > 0} \sin(\pi(\beta, H)) \tag{3}$$

where l is the number of positive roots. From the Weyl character formula $A_\rho(H)\chi_\lambda(H) = A_{\lambda+\rho}(H)$ it follows that

$$|\delta| = |\chi_\lambda(H)| = \left| \lim_{t \to 1} \frac{A_{\lambda+\rho}(tH)}{A_\rho(tH)} \right| \tag{4}$$

where $H = H_{i_0}/2$. For all $\tilde{H}$ such that $\tilde{H} = w(H_{i_0}/2)$, where $w \in W'_a$, it follows that $|\delta| = |\chi_\lambda(H_{i_0}/2)| = |\chi_\lambda(\tilde{H})|$. So we can give the following definition. The elements H_1, H_2 are called equivalent if there exists $s \in W$ such that $s(H_1) - H_2 \in P(R^\vee)$, and we shall write $H_1 \equiv H_2 (\mathrm{mod}\, P(R^\vee))$.

The foregoing proves the theorem.

Theorem 1. Let g_σ be a real form of simple complex algebra g, $\theta = \sigma \circ \tau = \exp(\mathrm{ad}(H_{i_0}/2))$, and χ_λ the character of the irreducible representation $\varphi: g \to sl(V)$.

Then $\varphi(g_\sigma) \subset su(p,q)$, and $\delta = p - q$ satisfies formula (4), where $H \equiv H_{i_0}/2$ $(\mathrm{mod}\, P(R^\vee))$.

4. The case $g = G_2$, $g_\sigma = G$ [2]

The extended Dynkin diagram for G_2 is

$$\overset{\alpha_1}{0} \Lleftarrow \overset{\alpha_2}{0} - 0$$

We shall take the roots realization from [6], that is $|\alpha_1| = \sqrt{2}$, $|\alpha_2| = \sqrt{6}$. The element $H = H_2/2$ defines automorphism $\theta = \exp(\mathrm{ad}\, H)$. Let ω_1, ω_2 be basis representations of G_2. This means that

$$\frac{2(\omega_i, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{i,k};\, i, k = 1, 2.$$

Then

$$H_2 = \frac{2\omega_2}{(\alpha_2, \alpha_2)} = \frac{\omega_2}{3}$$

$$H_1 = \frac{2\omega_1}{(\alpha_1, \alpha_1)} = \omega_1.$$

Furthermore

$$\frac{H_2}{2} \equiv \frac{H_2}{2} + H_1 + H_2 = \frac{\omega_2}{6} + \omega_1 + \frac{\omega_2}{3} = \frac{1}{2}(\rho + \omega_1)$$
$$= \tfrac{1}{2}(\rho + 2\alpha_1 + \alpha_2) \equiv \tfrac{1}{2}(\rho + \alpha_2) \equiv \tfrac{1}{2}(\rho - \alpha_2) \equiv \tfrac{1}{2}\rho (\mathrm{mod}\, P(R^\vee)).$$

Hence from (4) it follows that

$$|\delta| = \left| \lim_{t \to 1} \frac{A_{\lambda+\rho}(\frac{1}{2}t\rho)}{A_\rho(\frac{1}{2}t\rho)} \right| = \left| \lim_{t \to 1} \frac{A_\rho(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t\rho)} \right| \tag{5}$$

Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ be a highest weight of the representation φ. The limit in (4) depends on whether λ_1, λ_2 are even or odd. Thus from (3) and (5) table 1 is derived for the calculation of δ.

Table 1. The signatures $|\delta|$ of the representation $\overset{\lambda_1}{0} \Lleftarrow \overset{\lambda_2}{0}$ of G.

$\overset{\lambda_1}{0} \Lleftarrow \overset{\lambda_2}{0}$	$\lvert\delta\rvert$
$\overset{e}{0} \Lleftarrow \overset{e}{0}$	$\frac{1}{8}(\lambda_1 + 3\lambda_2 + 4)(\lambda_1 + \lambda_2 + 2)$
$\overset{e}{0} \Lleftarrow \overset{o}{0}$	$\frac{1}{8}(\lambda_2 + 1)(2\lambda_1 + 3\lambda_2 + 5)$
$\overset{o}{0} \Lleftarrow \overset{e}{0}$	$\frac{1}{8}(\lambda_1 + 1)(\lambda_1 + 2\lambda_2 + 3)$
$\overset{o}{0} \Lleftarrow \overset{o}{0}$	0

Symbol $e(o)$ in the column λ_i denotes an even (odd) λ_i.

5. The case $g=F_4$, $g_o=F\,II$

The extended Dynkin diagram for F_4 is

$$o-\overset{\alpha_1}{o}-\overset{\alpha_2}{o}\Rightarrow\overset{\alpha_3}{o}-\overset{\alpha_4}{o}$$

We shall take the roots realization from [6], that is $|\alpha_1|=|\alpha_2|=\sqrt{2}$, $|\alpha_3|=|\alpha_4|=1$. The element $H=H_4/2$ defines automorphism $\theta=\exp(\operatorname{ad} H)$. Let ω_1, ω_2, ω_3, ω_4 be basis representations of F_4. Then $H_1=\omega_1$, $H_2=\omega_2$, $H_3=2\omega_3$, $H_4=2\omega_4$. Furthermore

$$\frac{H_4}{2}\equiv\frac{H_4}{2}+H_1+H_2+H_3=\omega_4+\omega_1+\omega_2+2\omega_3=\rho+\omega_3$$

$$=\rho+2\alpha_1+4\alpha_2+6\alpha_3+3\alpha_4\equiv\rho+3\alpha_4\equiv\rho-\alpha_4\equiv\rho(\operatorname{mod} P(R^{\vee})).$$

Hence from (4) it follows that

$$|\delta|=\left|\lim_{t\to1}\frac{A_{\lambda+\rho}(t\rho)}{A_\rho(t\rho)}\right|=\left|\lim_{t\to1}\frac{A_\rho(t(\lambda+\rho))}{A_\rho(t\rho)}\right|. \tag{6}$$

Let

$$\lambda=\sum_{i=1}^{4}\lambda_i\omega_i$$

be the highest weight of the representation φ. The limit in (6) depends on whether λ_3, λ_4 are even or odd. From (3) and (6) table 2 is derived for the calculation of δ.

Table 2. The signatures $|\delta|$ of the representation $\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Rightarrow\overset{\lambda_3}{o}-\overset{\lambda_4}{o}$ of $F\,II$.

$\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Rightarrow\overset{\lambda_3}{o}-\overset{\lambda_4}{o}$	$\lvert\delta\rvert$
$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{o}{o}-\overset{o}{o}$	0
$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{e}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^4 5^2 7}A_1A_2A_5A_8A_9A_{10}A_{11}A_{12}A_{14}A_{16}A_{18}A_{19}A_{20}A_{22}A_{23}A_{24}$
$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{e}{o}-\overset{o}{o}$	$\frac{1}{2^{11}3^4 5^2 7}A_1A_2A_4A_5A_{11}A_{12}A_{13}A_{14}A_{15}A_{16}A_{17}A_{18}A_{20}A_{22}A_{23}A_{24}$
$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{o}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^4 5^2 7}A_1A_2A_3A_5A_6A_7A_{11}A_{12}A_{14}A_{16}A_{18}A_{20}A_{21}A_{22}A_{23}A_{24}$

Symbol e (o) in the column λ_i denotes an even (odd) λ_i. Symbol a denotes any λ_i independent of whether it is even or odd. The elements A_i, $i=1,\ldots,24$, must be taken from table 3.

Table 3. The elements A_i ($i=1,\ldots,24$) which are the scalar products $(\beta, \lambda+\rho)$, $\beta\in R$, $\beta>0$.

$A_1=\lambda_1+1$	$A_2=\lambda_2+1$	$A_3=\frac{1}{2}(\lambda_3+1)$
$A_4=\frac{1}{2}(\lambda_4+1)$	$A_5=\lambda_1+\lambda_2+2$	$A_6=\frac{1}{2}(2\lambda_2+\lambda_3+3)$
$A_7=\frac{1}{2}(2\lambda_1+2\lambda_2+\lambda_3+5)$	$A_8=\frac{1}{2}(2\lambda_1+2\lambda_2+\lambda_3+5)$	$A_9=\frac{1}{2}(2\lambda_1+2\lambda_2+\lambda_3+\lambda_4+6)$
$A_{10}=\frac{1}{2}(\lambda_3+\lambda_4+2)$	$A_{11}=\lambda_2+\lambda_3+2$	$A_{12}=\lambda_1+\lambda_2+\lambda_3+3$
$A_{13}=\frac{1}{2}(2\lambda_2+2\lambda_3+\lambda_4+5)$	$A_{14}=\lambda_1+2\lambda_2+\lambda_3+4$	$A_{15}=\frac{1}{2}(2\lambda_1+2\lambda_2+2\lambda_3+\lambda_4+7)$
$A_{16}=\lambda_2+\lambda_3+\lambda_4+3$	$A_{17}=\frac{1}{2}(2\lambda_1+4\lambda_2+2\lambda_3+\lambda_4+9)$	$A_{18}=\lambda_1+\lambda_2+\lambda_3+\lambda_4+4$
$A_{19}=\frac{1}{2}(2\lambda_1+4\lambda_2+3\lambda_3+\lambda_4+10)$	$A_{20}=\lambda_1+2\lambda_2+\lambda_3+\lambda_4+5$	$A_{21}=\frac{1}{2}(2\lambda_1+4\lambda_2+3\lambda_3+2\lambda_4+11)$
$A_{22}=\lambda_1+2\lambda_2+2\lambda_3+\lambda_4+6$	$A_{23}=\lambda_1+3\lambda_2+2\lambda_3+\lambda_4+7$	$A_{24}=2\lambda_1+3\lambda_2+2\lambda_3+\lambda_4+8$

Table 4. The signatures $|\delta|$ of the representation $\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Rightarrow\overset{\lambda_3}{o}-\overset{\lambda_4}{o}$ of FI.

$\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Rightarrow\overset{\lambda_3}{o}-\overset{\lambda_4}{o}$	$\lvert\delta\rvert$
$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{o}{o}-\overset{o}{o}$	0
$\overset{e}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_5A_8A_9A_{10}A_{11}A_{14}A_{18}A_{19}A_{22}A_{24}$
$\overset{e}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}-\overset{o}{o}$	$\frac{1}{2^{11}3^25}A_4A_5A_{11}A_{13}A_{14}A_{15}A_{16}A_{17}A_{20}A_{23}$
$\overset{e}{o}-\overset{e}{o}\Rightarrow\overset{o}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_3A_5A_6A_7A_{12}A_{16}A_{20}A_{21}A_{22}A_{24}$
$\overset{e}{o}-\overset{o}{o}\Rightarrow\overset{e}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_2A_8A_9A_{10}A_{12}A_{14}A_{16}A_{19}A_{22}A_{23}$
$\overset{e}{o}-\overset{o}{o}\Rightarrow\overset{e}{o}-\overset{o}{o}$	$\frac{1}{2^{11}3^25}A_2A_4A_{12}A_{13}A_{14}A_{15}A_{17}A_{18}A_{20}A_{24}$
$\overset{e}{o}-\overset{o}{o}\Rightarrow\overset{o}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_2A_3A_6A_7A_{11}A_{18}A_{20}A_{21}A_{22}A_{23}$
$\overset{o}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_1A_8A_9A_{10}A_{11}A_{12}A_{19}A_{20}A_{23}A_{24}$
$\overset{o}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}-\overset{o}{o}$	$\frac{1}{2^{11}3^25}A_1A_4A_{11}A_{12}A_{13}A_{15}A_{16}A_{17}A_{18}A_{22}$
$\overset{o}{o}-\overset{e}{o}\Rightarrow\overset{o}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_1A_3A_6A_7A_{14}A_{16}A_{18}A_{21}A_{23}A_{24}$
$\overset{o}{o}-\overset{o}{o}\Rightarrow\overset{e}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_1A_2A_5A_8A_9A_{10}A_{16}A_{18}A_{19}A_{20}$
$\overset{o}{o}-\overset{o}{o}\Rightarrow\overset{o}{o}-\overset{e}{o}$	$\frac{1}{2^{11}3^25}A_1A_2A_3A_5A_6A_7A_{11}A_{12}A_{14}A_{21}$
$\overset{o}{o}-\overset{o}{o}\Rightarrow\overset{e}{o}-\overset{o}{o}$	$\frac{1}{2^{11}3^25}A_1A_2A_4A_5A_{13}A_{15}A_{17}A_{22}A_{23}A_{24}$

The elements A_i, $i=1,\ldots,24$, have the same meaning as in table 2.

Table 5. The signatures $|\sigma|$ of the representation $\overset{\lambda_1}{o}\Rightarrow\overset{\lambda_2}{o}$ of $so_{2,3}$, $so_{1,4}$.

$\overset{\lambda_1}{o}\Rightarrow\overset{\lambda_2}{o}$	$\lvert\delta\rvert$ for $g_\sigma=so_{2,3}$	$\lvert\delta\rvert$ for $g_\sigma=so_{1,4}$
$\overset{a}{o}\Rightarrow\overset{o}{o}$	0	0
$\overset{e}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{2}(\lambda_1+\lambda_2+2)$	$\frac{1}{2}(\lambda_1+1)(\lambda_1+\lambda_2+2)$
$\overset{o}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{2}(\lambda_1+1)$	

Table 6. $g=so_7(\mathbb{C})$.

g_σ	$\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Rightarrow\overset{\lambda_3}{o}$	$\lvert\delta\rvert$
	$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{o}{o}$	0
	$\overset{e}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{16}(\lambda_1+\lambda_2+2)(\lambda_1+2\lambda_2+\lambda_3+4)(\lambda_2+\lambda_3+2)$
$so_{3,4}$	$\overset{o}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{16}(\lambda_1+1)(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_2+\lambda_3+2)$
	$\overset{e}{o}-\overset{o}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{16}(\lambda_2+1))(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_1+2\lambda_2+\lambda_3+4)$
	$\overset{0}{o}-\overset{0}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{16}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)$
	$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{o}{o}$	0
	$\overset{e}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{48}(2\lambda_1+\lambda_3+3)(\lambda_1+\lambda_2+2)(\lambda_1+2\lambda_2+\lambda_3+4)(\lambda_2+\lambda_3+2)$
$so_{2,5}$	$\overset{o}{o}-\overset{e}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{48}(2\lambda_1+4\lambda_2+\lambda_3+7)(\lambda_1+1)(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_2+\lambda_3+2)$
	$\overset{e}{o}-\overset{o}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{48}\lvert 2\lambda_1-\lambda_3+1\rvert(\lambda_2+1)(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_1+2\lambda_2+\lambda_3+4)$
	$\overset{o}{o}-\overset{o}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{48}(2\lambda_1+4\lambda_2+3\lambda_3+9)(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)$

Table 7. $g=so_7(\mathbb{C})$, $g_\sigma=so_{1,6}$.

$\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Rightarrow\overset{\lambda_3}{o}$	$\lvert\delta\rvert$
$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{o}{o}$	0
$\overset{a}{o}-\overset{a}{o}\Rightarrow\overset{e}{o}$	$\frac{1}{48}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)(\lambda_2+\lambda_3+2)(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_1+2\lambda_2+\lambda_3+4)$

Table 8. $g=sp_6(\mathbb{C})$, $g_\sigma=sp_{1,2}$.

$\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Leftarrow\overset{\lambda_3}{o}$	$\lvert\delta\rvert$
$\overset{e}{o}-\overset{e}{o}\Leftarrow\overset{a}{o}$	$\frac{1}{48}(\lambda_3+1)(\lambda_2+\lambda_3+2)(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_1+\lambda_2+2)(\lambda_1+\lambda_2+2\lambda_3+4)$
$\overset{e}{o}-\overset{o}{o}\Leftarrow\overset{a}{o}$	$\frac{1}{48}(\lambda_3+1)(\lambda_2+\lambda_3+2)(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_2+1)(\lambda_2+2\lambda_3+3)$
$\overset{o}{o}-\overset{e}{o}\Leftarrow\overset{a}{o}$	$\frac{1}{48}(\lambda_3+1)(\lambda_2+\lambda_3+2)(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_1+1)(\lambda_1+2\lambda_2+2\lambda_3+5)$
$\overset{o}{o}-\overset{o}{o}\Leftarrow\overset{a}{o}$	0

Table 9. $g=sp_6(\mathbb{C})$, $g_\sigma=sp_6(\mathbb{R})$.

$\overset{\lambda_1}{o}-\overset{\lambda_2}{o}\Leftarrow\overset{\lambda_3}{o}$	$\lvert\delta\rvert$
$\overset{e}{o}-\overset{e}{o}\Leftarrow\overset{e}{o}$	$\frac{1}{16}(\lambda_2+\lambda_3+2)(\lambda_1+\lambda_2+2)(\lambda_1+\lambda_2+2\lambda_3+4)$
$\overset{e}{o}-\overset{o}{o}\Leftarrow\overset{e}{o}$	$\frac{1}{16}(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_2+1)(\lambda_2+2\lambda_3+3)$
$\overset{o}{o}-\overset{e}{o}\Leftarrow\overset{o}{o}$	$\frac{1}{16}(\lambda_3+1)(\lambda_1+1)\cdot(\lambda_1+2\lambda_2+2\lambda_3+5)$
in other cases	0

Table 10. $g = so_8(\mathbb{C})$, $g_o = so_{4,4}$.

λ_1 λ_2 λ_3 o– o–o | λ_4o	$\lvert\delta\rvert$
e e e o– o–o | e o	$\frac{1}{32}(\lambda_1+\lambda_2+2)(\lambda_1+\lambda_2+\lambda_3+\lambda_4+4)(\lambda_2+\lambda_3+2)(\lambda_2+\lambda_4+2)$
e o e o– o–o | e o	$\frac{1}{32}(\lambda_1+\lambda_2+\lambda_3+3)(\lambda_2+1)(\lambda_2+\lambda_3+\lambda_4+3)(\lambda_1+\lambda_2+\lambda_4+3)$
o e o o– o–o | o o	$\frac{1}{32}(\lambda_1+1)(\lambda_3+1)(\lambda_4+1)(\lambda_1+2\lambda_2+\lambda_3+\lambda_4+5)$
in other cases	0

Symbols $e(o, a)$ have the same meaning in tables 1–10.

6. The case $g = F_4$, $g_\sigma = FI$

The element $H = H_1/2$ defines automorphism $\theta = \exp(\operatorname{ad} H)$. Similarly

$$\frac{H_1}{2} \equiv \frac{H_1}{2} + H_2 + H_3 + H_4 = \frac{1}{2}(\rho + \omega_2 + 3\omega_3 + 3\omega_4) \equiv \frac{1}{2}(\rho + \omega_3 + \omega_4 - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4) \equiv \tfrac{1}{2}(\rho + \omega_3 + \omega_4) (\operatorname{mod} P(R^{\vee})).$$

Note that

$$\rho + \omega_3 + \omega_4 = \rho^* = \frac{1}{2} \sum_{\beta^* \in R^{\vee}, \beta^* > 0} \beta^*.$$

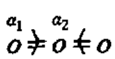
Hence from (4) it follows that

$$|\delta| = \left| \lim_{t \to 1} \frac{A_{\rho^*}(\frac{1}{2}t(\lambda + \rho))}{A_\rho(\frac{1}{2}t\rho^*)} \right|. \tag{7}$$

The limit in (7) depends on whether $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even or odd. From (3) and (7) we derive table 4 for the calculation of δ.

7. The case $g = so_5$ ($\mathbb{C}$)

The extended Dynkin diagram for B_2 is



We shall take the roots realization from [6], that is $|\alpha_1| = \sqrt{2}$, $|\alpha_2| = 1$. The element

$H=H_2/2$ defines automorphism $\theta=\exp(\operatorname{ad} H)$ for algebra $so_{1,4}$. Let ω_1, ω_2 be basis representations of B_2. Then $H_1=\omega_1$, $H_2=2\omega_2$. Furthermore

$$\frac{H_2}{2}\equiv\frac{H_2}{2}+H_1=\omega_2+\omega_1=\rho(\text{mod})\,PR^{\check{}})).$$

Hence from (4) it follows that

$$|\delta|=\left|\lim_{t\to 1}\frac{A_\rho(t(\lambda+\rho))}{A_\rho(t\rho)}\right|. \tag{8}$$

The element $H=H_1/2$ defines automorphism $\theta=\exp(\operatorname{ad} H)$ for algebra $so_{2,3}$. Similarly

$$\frac{H_1}{2}\equiv\frac{H_1}{2}+H_2=\frac{1}{2}\omega_1+2\omega_2=\frac{1}{2}(\rho+3\omega_2)\equiv\frac{1}{2}(\rho+\omega_2)(\text{mod}\,P(R^{\check{}})).$$

Note that

$$\rho+\omega_2=\rho^*=\frac{1}{2}\sum_{\beta^*\in R^{\check{}},\beta^*>0}\beta^*.$$

Hence from (4) it follows that

$$|\delta|=\left|\lim_{t\to 1}\frac{A_{\rho^*}(\frac{1}{2}t(\lambda+\rho))}{A_\rho(\frac{1}{2}t\rho^*)}\right|. \tag{9}$$

Let $\lambda=\lambda_1\omega_1+\lambda_2\omega_2$ be the highest weight of the representation φ. The limits in (8) and (9) depend on whether λ_1, λ_2 are even or odd. Similarly, from (8) and (9) we derive table 5 for the calculation of δ.

8. The cases $g=so_7(\mathbb{C})$, $so_8(\mathbb{C})$, $sp_6(\mathbb{C})$

Discussing this in the same way, we obtain tables 6–10 for the calculation of $|\delta|$.

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